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EFFECT OF INITIAL CONDITIONS ON TRAVELING WAVE TUBES

by

WILLIAM SOLLFREY

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Abstract

The effect of the excitation conditions of a traveling wave tube on its operation is analyzed in detail. By taking the Laplace transform of the field equations with respect to the longitudinal coordinate, the initial conditions are brought directly into the equations, so there is no problem of matching at the input. The transformed equations may be solved, and the actual fields and currents then determined by inverting the transform. The current in the amplified mode is found and it is expressed in terms of transfer coefficients which give the current produced down the tube by arbitrary distributions of input field, input current, or input a.c. velocity. In this paper, only the analytical work is presented. Applications to specific excitation systems will appear in a second paper.

I. Introduction

Previous research[1,2] on the traveling wave tube have proceeded along the following lines. First the helical portion of the tube is assumed to be infinitely long. The physical helix is replaced by a cylindrical sheath which is conducting only in the helical direction. The properties of the three discrete modes of this system are then studied in detail. To find how these modes are excited by the input fields of the actual physical system; the input electric field, electron velocity, and a.c. charge density are then expressed in terms of the coefficients of the three modes, and the resulting equations solved for these coefficients. This theory has been extended phenomenologically by introducing a parameter into the equations to take account of passive modes[3], but again the fields of the finite length helix are replaced by those of the infinitely long system. In this paper, we shall study the validity of these processes and present a new method of treating the traveling wave tube which surmounts the difficulties of the previous analysis.

The first difficulty faced by the earlier analysis is that the discrete modes of the helical sheath system are not complete. Consequently, it is not possible to expand an arbitrary input field distribution in terms of them, since we can always find field distributions which are orthogonal to the space configurations of all the discrete modes and hence have vanishing expansion coefficients. It is generally assumed that this effect is avoided theoretically by making the beam small and neglecting radial variation of the fields. However, at least one component of the fields must fall to zero at the helix, while other components are strong near the helix, so the approximation may be expected to be considerably in error. Since there are no other discrete modes, to obtain a complete system we shall include the continuous spectrum of the helical sheath system. This continuous spectrum does not appear to have been discussed previously.

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1. J. R. Pierce, "Theory of the Beam Traveling Wave Tube", P.I.R.E. 35, 111 (1947).
2. L. J. Chu and J. D. Jackson, "Field Theory of Traveling Wave Tubes", P.I.R.E., 36, 853 (1948).
3. J. R. Pierce, "Effect of Passive Modes in TWT's", P.I.R.E., 36, 993, (1948).

A second difficulty arises from the assumption that the fields are circularly symmetrical. The traveling wave tube is generally excited by having the beam cross a waveguide, and the fields of the waveguide are not circularly symmetrical in the system of cylindrical coordinates based on the beam. Hence, components of field having angular dependence may be generated. This possibility is included in the analysis to be presented.

The method to be used is founded on the premise that the incident fields in the input waveguide are strong compared to the fields generated by the beam in the waveguide. Consequently, we shall neglect the reaction of the beam upon the input fields. As a result, we may assume that the fields and the electron velocity and current are given functions of the transverse coordinates at the place where the beam enters the helical section. We also assume that the helix is sufficiently long that the backward waves generated at the output section may be neglected near the input. Thus, we shall treat the helix as semi-infinite in length, with the input fields specified at $z = 0$. The question to be treated is: given the input fields, to find the fields along the helix and especially to find the strength of the amplified wave.

A natural method for treating semi-infinite systems is to apply the Laplace transformation. This approach will bring in the initial conditions in a direct manner. The fields will be expressed as contour integrals, whose integrands will possess both poles and branch points. The residues at the poles will represent the discrete modes of the system and the integrals around the necessary branch cuts will provide the continuous spectrum. The theory may be developed for arbitrary angular dependence with no greater complications than arise in the symmetric case.

We shall determine the current in the beam at large distances down the helix. Since the radius of the hole in the wall of the output waveguide through which the beam passes is a small fraction of the wavelength in the guide, most of the field in the helical section is reflected, and the major coupling to the output guide is by means of the a.c. beam current. This current is expressed in terms of transfer coefficients which represent the current that would be generated by unit field, unit charge, or unit a.c. electron velocity.

In the next section we transform and solve the field equations. The matching conditions at the beam and at the helix are applied to the solutions in Section 3, where we eliminate the arbitrary constants that appear. In Section 4 we invert the transforms and interpret the results. In this paper the analytical results are presented. Application to practical exciting systems and some computations of experimentally determinable quantities will appear in a second paper.

2. Solution of the Field Equations

In order to apply Laplace transform theory we wish to have the properties of the system independent of the z coordinate. Consequently, we replace the physical helix by a lossless cylindrical sheath of radius a which is conducting in the helical direction only. This approximation has been customarily employed in traveling wave tube research. We assume that a strong longitudinal static magnetic field constrains the electrons to have only longitudinal motion. The lifting of this assumption will be discussed in a future paper. The electron beam is assumed to be cylindrical, of radius b, and the d.c. electron velocity is assumed to be constant across the beam. This approximation limits the d.c. beam current, else the d.c. potential will fall near the beam axis. We limit ourselves to a small-signal analysis, and therefore assume that a.c. quantities are small compared to the associated d.c. quantities. This approximation linearizes the equations and makes a solution possible. We use cylindrical coordinates with the z-axis along the beam axis.

The system is governed by nine equations; that is, six Maxwell equations, the equation of continuity, the relation between current, charge, and velocity, and the force equation for the electrons. We separate d.c. and a.c. quantities, using the time factor $e^{i\omega t}$ for the latter, and keep only first order terms in the a.c. quantities. The equations become:

2.1)
$$\begin{aligned}
\nabla \times \underrightarrow{E}' + j\omega\mu \underrightarrow{H}' &= 0 \\
\nabla \times \underrightarrow{H}' - j\omega\epsilon \underrightarrow{E}' &= \underrightarrow{J}' \\
\nabla \cdot \underrightarrow{J}' + j\omega\rho' &= 0 \\
\underrightarrow{J}' &= \rho' v_o \underrightarrow{i}_z + \rho_o \underrightarrow{v}' \\
m\left(j\omega + v_o \frac{\partial}{\partial z}\right) \underrightarrow{v}' &= e\underrightarrow{E}'
\end{aligned}$$

All primed quantities are a.c., while all quantities with subscript zero are d.c. We have assumed the d.c. electron velocity to be much less than the velocity of light, and hence have neglected the interaction of the electrons with the a.c. magnetic field. Because of the strong d.c. magnetic field, the velocity has only a z component, and the same is true of the current. We may save much writing by introducing a system of reduced units.

We thus measure $\underline{v}'$ in units of v_0, ρ' in units of ρ_0, $\underline{J}'$ in units of $J_0 = \rho_0 v_0$, $\underline{E}'$ in units of $2V_0$, where V_0 is the beam voltage, and $\underline{H}'$ in units of $2V_0\sqrt{\epsilon/\mu}$. Thus:

$$\text{2.2)} \qquad \begin{aligned} &v_z' = v_0 v_z \,, \qquad \rho' = \rho_0 \rho \,, \qquad J_z' = \rho_0 v_0 \, J_z \,, \\ &\underline{E}' = 2V_0 \underline{E} \,, \qquad \underline{H}' = 2V_0 \sqrt{\epsilon/\mu}\, \underline{H} \,. \end{aligned}$$

We introduce some associated symbols:

$$\text{2.3)} \qquad \begin{aligned} &k = \omega/c, \qquad\qquad k' = \omega/v_0 \,, \qquad\qquad \beta = v_0/c \,, \\ &h^2 = \rho_0/2\epsilon V_0 = J_0/2\epsilon v_0 V_0 = J_0/2\epsilon\sqrt{2e/m}\; V_0^{3/2} \;. \end{aligned}$$

The quantity h is the debunching wave number familiar from klystron theory. It is a measure of the strength of the d.c. space charge forces. The assumption that the d.c. beam velocity is constant across the beam implies hb is small compared to unity. In these reduced units, the system of equations becomes:

$$\nabla \times \underline{E} + jk\underline{H} = 0$$

$$\nabla \times \underline{H} - jk\underline{E} = \beta h^2(\rho + v_z)\, \underline{i}_z$$

$$\text{2.4)} \qquad \frac{\partial v_z}{\partial z} + \left(\frac{\partial}{\partial z} + jk'\right)\rho = 0$$

$$J_z = \rho + v_z$$

$$\left(\frac{\partial}{\partial z} + jk'\right) v_z = E_z$$

We assume that all quantities have the angular dependence $e^{jn\theta}$. The results may be summed over n at the end of the analysis. The next step is to take the Laplace transform with respect to z of the set of equations 2.4). For any quantity F, we define the Laplace transform $\overline{F}$ by:

$$\text{2.5)} \qquad \overline{F}(r,\theta,s) = \int_0^\infty F(r,\theta,z)e^{-sz}\, dz$$

The transform of a derivative is given by:

$$\text{2.6)} \qquad \int_0^\infty \frac{\partial F(r,\theta,z)}{\partial z}\, e^{-sz}\, dz = -F(r,\theta,0) + s\overline{F}(r,\theta,s)$$

We shall find the solutions of the transformed equations. We then interpret the results by using the inverse transform:

$$2.7) \qquad F(r,\theta,z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{F}(r,\theta,s)\, e^{sz}\, ds \; .$$

The contour lies to the right of all singularities of the integrand.

The result of transforming the equations 2.4) is the more manageable set of equations:

$$2.8) \qquad \begin{aligned}
&\frac{jn}{r}\bar{E}_z - s\bar{E}_\theta + jk\bar{H}_r = -E_\theta(0) \\
&s\bar{E}_r - \frac{\partial \bar{E}_z}{\partial r} + jk\bar{H}_\theta = E_r(0) \\
&\frac{jn}{r}\bar{H}_z - s\bar{H}_\theta - jk\bar{E}_r = -H_\theta(0) \\
&s\bar{H}_r - \frac{\partial \bar{H}_z}{\partial r} - jk\bar{E}_\theta = H_r(0) \\
&\frac{1}{r}\frac{\partial}{\partial r} r\,\bar{E}_\theta - \frac{jn}{r}\bar{E}_r + jk\bar{H}_z = 0 \\
&\frac{1}{r}\frac{\partial}{\partial r} r\,\bar{H}_\theta - \frac{jn}{r}\bar{H}_r - jk\bar{E}_z = \beta h^2(\bar{\rho} + \bar{v}_z) \\
&s\bar{v}_z + (s + jk')\bar{\rho} = \rho(0) + v_z(0) \\
&\bar{J}_z = \bar{\rho} + \bar{v}_z \\
&(s + jk')\,\bar{v}_z = \bar{E}_z + v_z(0)
\end{aligned}$$

The argument 0 refers to $z = 0$, while all barred quantities are functions of r and s, with the θ dependence $e^{jn\theta}$ implied. We see that the initial conditions have come into the equations in a perfectly natural manner, without the necessity of matching them to a modal decomposition of the fields.

The first four equations of 2.8) may be solved for the transverse components, $\bar{E}_r$, $\bar{E}_\theta$, $\bar{H}_r$, $\bar{H}_\theta$, in terms of the longitudinal components $\bar{E}_z$, $\bar{H}_z$, and the incident field components. Similarly, the last three equations may be solved for the beam quantities $\bar{v}_z$, $\bar{\rho}$, and $\bar{J}_z$ in terms of $\bar{E}_z$

and the incident electronic components. The solutions of these seven equations are:

2.9)

$$\bar{E}_r = \frac{1}{s^2+k^2}\left[s\,\frac{\partial \bar{E}_z}{\partial r} + \frac{nk}{r}\,\bar{H}_z + sE_r(0) - jk\,H_\theta(0)\right]$$

$$\bar{E}_\theta = \frac{1}{s^2+k^2}\left[\frac{jns}{r}\,\bar{E}_z + jk\,\frac{\partial \bar{H}_z}{\partial r} + sE_\theta(0) + jkH_r(0)\right]$$

$$\bar{H}_r = \frac{1}{s^2+k^2}\left[-\frac{nk}{r}\,\bar{E}_z + s\,\frac{\partial \bar{H}_z}{\partial r} + sH_r(0) + jkE_\theta(0)\right]$$

$$\bar{H}_\theta = \frac{1}{s^2+k^2}\left[-jk\,\frac{\partial \bar{E}_z}{\partial r} + \frac{jns}{r}\,\bar{H}_z + sH_\theta(0) - jkE_r(0)\right]$$

$$\bar{v}_z = \frac{1}{s + jk'}\left[\bar{E}_z + v_z(0)\right]$$

$$\bar{\rho} = \frac{1}{(s+jk')^2}\left[-s\bar{E}_z + jk'v_z(0) + (s + jk')\,\rho(0)\right]$$

$$\bar{J}_z = \frac{1}{(s+jk')^2}\left[jk'\bar{E}_z + (s+2jk')\,v_z(0) + (s+jk')\,\rho(0)\right]$$

The solutions 2.9) may be inserted into the two remaining equations of 2.8). We define the "radial propagation constants" p and γ by the relations:

2.10)

$$p^2 = -(s^2 + k^2)$$

$$\gamma^2 = p^2\left[1 + \frac{h^2}{(s+jk')^2}\right]$$

The terms involving the incident fields may be considerably simplified by using the Maxwell's equation 2.4). The transverse input fields may be eliminated, and the equations expressed in terms of the longitudinal input fields and their derivatives. The resulting wave equations are:

2.11)

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\,r\,\frac{\partial}{\partial r} - p^2 - \frac{n^2}{r^2}\right]\bar{H}_z = \left(\frac{\partial}{\partial z} + s\right)H_z(0)$$

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\,r\,\frac{\partial}{\partial r} - \gamma^2 - \frac{n^2}{r^2}\right]\bar{E}_z = \left(\frac{\partial}{\partial z} + s\right)E_z(0) + \frac{h^2}{jk'}\left[-\frac{(k^2+k'^2)s+2jk^2k'}{(s+jk')^2}\,v_z(0) + \frac{k'^2-k^2}{s+jk'}\,\rho(0)\right].$$

The second of these equations holds only for $r < b$. When $r > b$, so we are outside the beam, the input velocity and charge density are zero, and γ is replaced by p.

We now wish to solve the wave equations 2.11) regarding the right-hand-sides as forcing terms, which are given functions of position. Boundary conditions are applied at the beam radius $r = b$ and the sheath radius $r = a$. The exact boundary conditions to be used will be stated in Section 3. To solve the equations, we introduce the Green's functions:

2.12)
$$
\begin{aligned}
G_1 &= I_n(\gamma r)\, K_n(\gamma r') && r < r' \\
&\quad K_n(\gamma r)\, I_n(\gamma r') && r > r' \\
G_2 &= I_n(pr)\, K_n(pr') && r < r' \\
&\quad K_n(pr)\, I_n(pr') && r > r'
\end{aligned}
$$

We are assuming that $K_n(pr)$ is that solution of the homogeneous wave equation which is exponentially decreasing at infinity. This requires that p have a positive real part. However, p is defined in terms of the complex variable s, and s lies on a vertical contour in the right half-plane. As a function of s, p has branch points at $s = jk$, $s = -jk$. The branch cuts may be so placed that p actually does have a positive or zero real part along the entire path of integration in the s-plane. The details of this analysis will be presented in Section 4.

We denote the right-hand sides of Equations 2.11) by F_M and F_E+F_J, respectively. Applying Green's theorem to the functions $\bar{E}_z, \bar{H}_z$, and the appropriate Green's functions yields the solutions of 2.11), each in its appropriate region of space,

2.13)
$$
\begin{aligned}
\bar{E}_{zi} &= A_1 I_n(\gamma r) - \int_0^b r'dr' G_1(r,r')\left[F_E(r') + F_J(r')\right] && r < b \\
\bar{E}_{zm} &= A_2 I_n(pr) + A_3 K_n(pr) - \int_b^a r'dr' G_2(r,r')\, F_E(r') && b < r < a \\
\bar{E}_{zo} &= A_4 K_n(pr) - \int_a^\infty r'dr' G_2(r,r')\, F_E(r') && r > a \\
\bar{H}_{zi,m} &= B_1 I_n(pr) - \int_0^a r'dr' G_2(r,r') F_M(r') && r < a \\
\bar{H}_{zo} &= B_2 K_n(pr) - \int_a^\infty r'dr' G_2(r,r')\, F_M(r') && r > a
\end{aligned}
$$

The letters i,m and o, stand for inner region $r < b$, middle region $b < r < a$, and outer region $r > a$, respectively. Since the longitudinal magnetic field is unaffected by the presence of the beam, we may use the same solution for $\overline{H}_z$ in regions i and m. The integrals appearing here represent particular integrals of the equations 2.11) due to the forcing terms, while we add appropriate solutions of the homogeneous equations to satisfy the boundary conditions. The conditions of regularity at $r = 0$ and $r = \infty$ are already satisfied by the solutions 2.13). We shall now use the boundary conditions at the beam radius and the sheath radius to eliminate the arbitrary constants $A_1, \ldots B_2$. This reduction will form the subject matter of the next section.

3. Application of the Boundary Conditions

The boundary conditions at the beam radius are that the tangential electric and magnetic fields shall be continuous. The second condition is a consequence of the presence of the infinitely strong static magnetic field. If the static field were finite, radial spreading of the beam would occur, and this would have to be replaced by a surface current density at the beam surface. This surface current would cause a discontinuity in the tangential magnetic field. However, this effect is suppressed by the infinite static field. Since the boundary conditions are to be satisfied for all z and all θ, the transformed field components must also satisfy the conditions. The continuity of $\bar{H}_z$ is already insured by the solutions 2.13). Hence, the boundary conditions to be applied at the beam are:

3.1)
$$\left.\begin{aligned} \bar{E}_{zi} &= \bar{E}_{zm} \\ \bar{E}_{\theta i} &= \bar{E}_{\theta m} \\ \bar{H}_{\theta i} &= \bar{H}_{\theta m} \end{aligned}\right\} \quad r = b$$

In view of the relations 2.9), these conditions can be replaced by the equivalent set of conditions:

3.2)
$$\left.\begin{aligned} \bar{E}_{zi} &= \bar{E}_{zm} \\ \frac{\partial}{\partial r}\bar{E}_{zi} &= \frac{\partial}{\partial r}\bar{E}_{zm} \end{aligned}\right\} \quad r = b$$

At the helical sheath, the boundary conditions are that the component of electric field in the helical direction shall vanish, the component of electric field upon the sheath but normal to the helical direction shall be continuous, and the component of magnetic field in the helical direction shall be continuous. The transformed components must also satisfy these conditions. The helix direction makes the angle π/2 - α with the z-axis, whence these conditions are expressed as:

3.3)
$$\left.\begin{aligned} &\bar{E}_{\theta m}\cos\alpha + \bar{E}_{zm}\sin\alpha = 0 \\ &\bar{E}_{\theta o}\cos\alpha + \bar{E}_{zo}\sin\alpha = 0 \\ &\bar{E}_{\theta m}\sin\alpha - \bar{E}_{zm}\cos\alpha = \bar{E}_{\theta o}\sin\alpha - \bar{E}_{zo}\cos\alpha \\ &\bar{H}_{\theta m}\cos\alpha + \bar{H}_{zm}\sin\alpha = \bar{H}_{\theta o}\cos\alpha + \bar{H}_{zo}\sin\alpha \end{aligned}\right\} \quad r = a$$

Upon using the relations 2.9), these conditions may be simplified slightly, yielding:

$$
3.4)\quad
\begin{aligned}
&\bar{E}_{zm} = \bar{E}_{zo} \\
&\frac{\partial}{\partial r}\,\bar{H}_{zm} = \frac{\partial}{\partial r}\,\bar{H}_{zo} \\
&\bar{E}_{\theta m}\cos\alpha + \bar{E}_{zm}\sin\alpha = 0 \\
&\bar{H}_{\theta m}\cos\alpha + \bar{H}_{zm}\sin\alpha = \bar{H}_{\theta o}\cos\alpha + \bar{H}_{zo}\sin\alpha
\end{aligned}
$$

Thus equations 3.2) and 3.4) provide six conditions to be satisfied by the solutions 2.13). These solutions contain six arbitrary constants, and since the forcing terms render the boundary condition equations inhomogeneous, a solution can be found for the arbitrary constants if the determinant of the equations does not vanish. Since the determinant will involve Bessel functions of the complex variable p, we may expect that the determinant will vanish only for certain values of p, if any. These critical values will correspond to the discrete modes of the system, which can be set up without forcing terms. Thus, the determinant does not vanish identically.

The boundary condition equations may be simplified by introducing a set of "transfer coefficients". These refer to the values of the integrals appearing in 2.13) when r = a or b, and will represent the effect of the initial conditions. The six coefficients we introduce are defined by:

$$
3.5)\quad
\begin{aligned}
C_1 &= \int_0^b r\,dr\, I_n(\eta r)\left[F_E(r) + F_J(r)\right] \\
C_2 &= \int_b^a r\,dr\, K_n(pr)\, F_E(r) \\
C_3 &= \int_b^a r\,dr\, I_n(pr)\, F_E(r) \\
C_4 &= \int_a^\infty r\,dr\, K_n(pr)\, F_E(r) \\
D_1 &= \int_0^a r\,dr\, I_n(pr)\, F_M(r) \\
D_2 &= \int_a^\infty r\,dr\, K_n(pr)\, F_M(r)
\end{aligned}
$$

The integrals that appear here will all converge if F_E and F_M are sufficiently regular at infinity, and this condition is always satisfied for practical exciting systems. We also introduce a constant μ, defined by:

$$\mu = p^2 \tan \alpha - jn\, s/a \, . \tag{3.6}$$

The six boundary condition equations may now be written as:

$$\begin{aligned}
&A_1 I_n(\gamma b) - A_2 I_n(pb) - A_3 K_n(pb) = C_1 K_n(\gamma b) - C_2 I_n(pb)\\
&\gamma b A_1 I_n'(\gamma b) - pbA_2 I_n'(pb) - pbA_3 K_n'(pb) = \gamma b\, C_1 K_n'(\gamma b) - pb\, C_2\, I_n'(pb)\\
&A_2 I_n(pa) + A_3 K_n(pa) - A_4 K_n(pa) = C_3 K_n(pa) - C_4 I_n(pa)\\
&B_1 I_n'(pa) - B_2 K_n'(pa) = D_1 K_n'(pa) - D_2 I_n'(pa)\\
&-\mu\left[A_2 I_n(pa) + A_3 K_n(pa)\right] + jkpB_1 I_n'(pa) = -\mu C_3 K_n(pa) + jkpD_1 K_n'(pa) - (sE_\theta(0,a) + jkH_r(0,a))\\
&jkp\left[A_2 I_n'(pa) + A_3 K_n'(pa) - A_4 K_n'(pa)\right] + \mu\left[B_1 I_n(pa) - B_2 K_n(pa)\right] =\\
&jkp\left[C_3 K_n'(pa) - C_4 I_n'(pa)\right] + \mu\left[D_1 K_n(pa) - D_2 I_n(pa)\right]
\end{aligned} \tag{3.7}$$

The prime denotes the derivative of a Bessel function with respect to its argument.

The quantity which is to be determined is the current in the beam. From the solutions 2.9), the transformed current in the beam is proportional to the transformed longitudinal electric field $\bar{E}_z$. Consequently, the constant which is to be solved for is A_1, which measures the strength of the electric field in the beam. However, it proves most convenient to solve equations 3.7) for A_2, and then express A_1 in terms of A_2. We shall proceed to solve equations 3.7) in sequence, rather than by using determinants, since for a sixth-order system such as 3.7) the determinants become too unwieldy.

First, we solve the third equation of 3.7) for A_4, and then solve the fourth equation for B_2. This expresses the constants describing the fields outside the helical sheath in terms of the constants for for the region inside the sheath. There results:

$$\begin{aligned}
A_4 &= (A_2 + C_4)\,\frac{I_n(pa)}{K_n(pa)} + (A_3 - C_3)\\
B_2 &= (B_1 + D_2)\,\frac{I_n'(pa)}{K_n'(pa)} - D_1
\end{aligned} \tag{3.8}$$

Next, these two relations are inserted into the last of equations 3.7). The resulting equation may be solved for B_1 in terms of A_2 and A_3. We thereby obtain the magnetic field constants in terms of the electric field constants. The result may be reduced by the Wronskian relationship:

3.9)
$$I_n'(z)\, K_n(z) - I_n(z)\, K_n'(z) = 1/z$$

and the solution becomes:

3.10)
$$B_1 = - D_2 + \frac{jkp}{\mu} \frac{K_n'(pa)}{K_n(pa)} (A_2+C_4)$$

This may be reinserted into 3.8) to express the outer magnetic constant B_2 in terms of electric field constants, yielding:

3.11)
$$B_2 = -D_1 + \frac{jkp}{\mu} \frac{I_n'(pa)}{K_n(pa)} (A_2+C_4)$$

Thus, to this point the outer electric field and the entire magnetic field have been eliminated, and we are left with three equations for the inner and middle electric field.

The third step in the reduction is to eliminate the constants A_1 and A_3 by solving the first and the second of equations 3.7). We introduce the symbol:

3.12)
$$\delta = \eta b\, I_n'(\eta b)\, K_n(pb) - pb\, I_n(\eta b)\, K_n'(pb)$$

This quantity δ is the determinant of the first two equations of 3.7). It reduces to unity when the beam current goes to zero, so that $h = 0$ and $\eta = p$. The solutions for A_1 and A_3 are found to be:

3.13)
$$A_1 = \frac{1}{\delta} \left[A_2-C_2 + C_1\{\eta b\, K_n'(\eta b)K_n(pb) - pb\, K_n'(pb)\, K_n(\eta b)\} \right]$$
$$A_3 = \frac{1}{\delta} \left[-C_1 + (A_2-C_2)\{ pb\, I_n'(pb)I_n(\eta b)- \eta b\, I_n'(\eta b)I_n(pb)\}\right]$$

It is assumed here that δ does not vanish. Since for large values of Re s the quantity η as defined by 2.10) is equal to $p + 0(1/s)$, for such values $\delta = 1 + 0(1/s)$. Consequently we may select a contour of integration such that δ does not vanish anywhere along it.

The penultimate step is to take the relations 3.8), 3.10), 3.11), and 3.13); and substitute all of them into the fifth of equations 3.7). The only remaining undetermined constant is A_2. This resulting equation may be simplified at many points by use of the Wronskian relation 3.9), and several other terms cancel. We define the function:

$$3.14) \qquad G_n(pa) = \frac{I_n(pa)K_n(pa) + \frac{(kp)^2}{\mu^2} I_n'(pa)K_n'(pa)}{K_n^2(pa)}$$

The solution for A_2 may be expressed in the form:

$$3.15) \qquad A_2 = N_2/M_2$$

$$M_2 = G_n(pa) + \left\{ pbI_n'(pb)\ I_n(\gamma b) - \gamma bI_n'(\gamma b)\ I_n(pb)\right\} /\delta$$

$$N_2 = C_1/\delta + C_2\left\{ pbI_n'(pb)\ I_n(\gamma b) - \gamma bI_n'(\gamma b)\ I_n(pb)\right\} /\delta$$

$$+ C_3 + \left\{I_n(pa) - C_n(pa)K_n(pa)\right\} C_4/K_n(pa)$$

$$- \frac{jkp}{\mu} \frac{K_n'(pa)}{K_n(pa)} D_1 - j\frac{kp}{\mu} \frac{I_n'(pa)}{K_n(pa)} D_2 + \frac{1}{\mu K_n(pa)}\left\{sE_\theta(0,a) + jkH_r(0,a)\right\}$$

The final step is to substitute this result into the first of equations 3.13) to obtain A_1. Only the coefficients of C_1 and C_2 in N_2 will be affected. After some reduction, the final result for A_1 is:

$$3.16) \quad A_1 = N_1/M_1$$

$$M_1 = pb\ I_n(\gamma b)\left\{ I_n'(pb) - G_n(pa)\ K_n'(pb)\right\}$$

$$- \gamma b\ I_n'(\gamma b)\left\{ I_n(pb) - G_n(pa)\ K_n(pb)\right\}$$

$$N_1 = \begin{bmatrix} pb\ K_n(\gamma b)\left\{ I_n'(pb) - G_n(pa)\ K_n'(pb)\right\} \\ - \gamma bK_n'(\gamma b)\left\{I_n(pb) - G_n(pa)\ K_n(pb)\right\} \end{bmatrix} C_1$$

$$- G_n(pa)C_2 + C_3 + \left\{I_n(pa) - G_n(pa)\ K_n(pa)\right\} C_4/K_n(pa)$$

$$- j\frac{kp}{\mu} \frac{K_n'(pa)}{K_n(pa)} D_1 - j\frac{kp}{\mu} \frac{I_n'(pa)}{K_n(pa)} D_2 + \frac{1}{\mu K_n(pa)} (sE_\theta(0,a) + jkH_r(0,a))$$

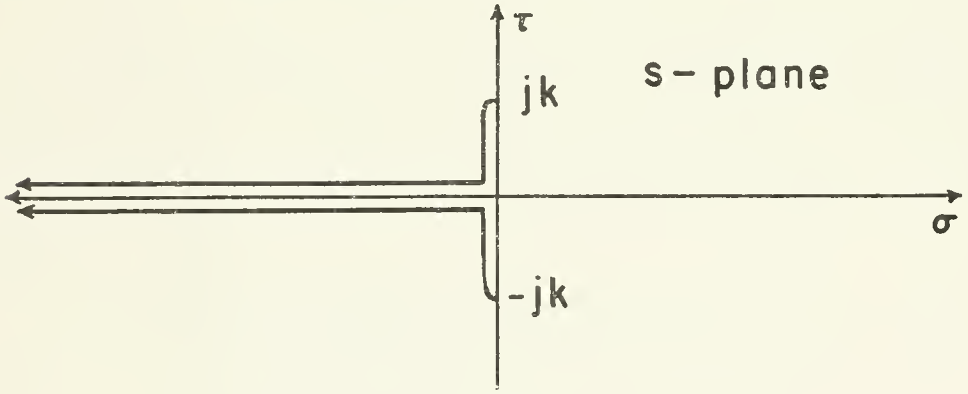
With this determination of the constant A_1, the transformed electric field $\overline{E}_z$ is known completely. The last of equations 2.9) now says that the transformed current in the beam is found by multiplying $\overline{E}_z$ by $jk'/(s+ jk')^2$ and adding some initial terms. The next step is to invert the Laplace transformation to obtain the actual current and then interpret the results.

4. Results and Interpretations

We now shall invert the Laplace transformation. The first steps in this process are to properly locate in the complex s-plane the contour of integration and any branch cuts that may be required. The contour must be placed to the right of all singularities, so we must study where the singularities are located.

The first step is to demonstrate that we can actually satisfy the requirement that the real part of p is to be positive. The definition $p^2 = -(s^2+k^2)$ shows that p has branch points at $s = \pm jk$. Since the contour of integration is to lie in the right half-plane, the branch cuts which must be inserted to render p single valued cannot extend to $+\infty$. We write $s = \sigma \pm i\tau$. By its definition, p is purely imaginary on the entire real axis, $\tau = 0$. It is also purely imaginary on the imaginary axis $\sigma = 0$ when $|\tau| < k$, and is purely real on the imaginary axis when $|\tau| > k$. The requirement is that $|\arg p| \leq \pi/2$, so that the K function either is exponentially small or an outgoing wave when r approaches infinity:

We shall place the branch cuts as in Figure 1.



The Branch Cuts for p
Figure 1

The upper branch cut starts from $s = jk$, goes along the imaginary axis to zero, remaining in the second quadrant, and then passes to minus infinity along the upper side of the negative real axis. The lower branch cut is a mirror image of the upper, and it lies wholly in the third quadrant.

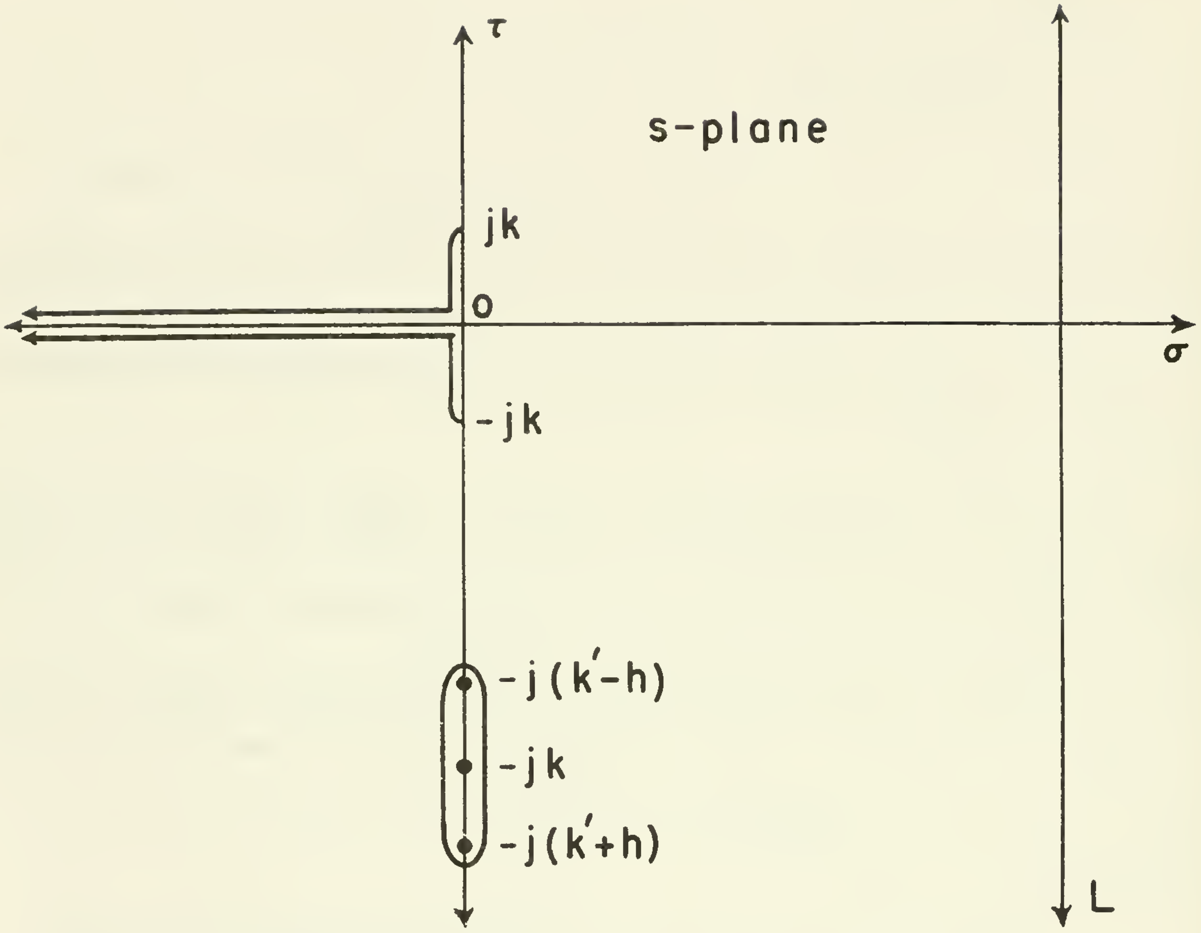
We now define the square root involved in p by:

4.1) $$\begin{aligned} p &= e^{j\frac{\pi}{2}}\sqrt{s^2+k^2} \qquad \tau = \mathrm{Im}\, s > 0 \\ &= e^{-j\frac{\pi}{2}}\sqrt{s^2+k^2} \qquad \tau = \mathrm{Im}\, s < 0 \end{aligned}$$

The square root appearing here is so taken that $\lim_{s\to 0}\sqrt{s^2+k^2} = k$, and it is real positive along both branch cuts. With this definition, p is positive imaginary along the upper branch cut, has a phase between zero and $\pi/2$ in the second quadrant, and is positive real along the upper section $\tau > k$ of the positive imaginary axis. The phase of p lies between zero and $-\pi/2$ in the first quadrant, and p is negative imaginary on the upper side of the positive real axis.

In the lower half plane, p is negative imaginary on the lower branch cut. Its phase is between zero and $-\pi/2$, in the third quadrant, and p is positive real along the lower section $-\tau > k$ of the negative imaginary axis. In the fourth quadrant, the phase of p is between zero and $\pi/2$, and p is positive imaginary on the lower side of the positive real axis. There accordingly is a jump in the value of p upon crossing the real axis. However, this jump does not contribute to the value of an integral. We see that with the branch cuts of Figure 1 and the definition 4.1) we actually have satisfied the requirement $|\arg p| \leq \pi/2$.

Other branch points of the integrand occur at the zeroes of η . By the definition 2.10), η has branch points at $s = \pm jk$ arising from the zeroes of p. It also has branch points at $s = -j(k' \pm h)$, and a simple pole at $s = -jk'$. We shall dispose of these latter singularities by a closed contour which surrounds the three singular points. The phase of the square root of the second factor in η^2 is specified by requiring that $\left[1 + h^2/(s + jk')^2\right]^{1/2}$ approaches 1 uniformly with respect to arg s as $|s|$ approaches infinity. This condition may be satisfied by the closed contour branch cut. Accordingly, the path of integration, L, and the branch cuts are given as in Figure 2



The Branch Cuts for η
Figure 2

The integrand has branch points at $-j(k'-h)$ and $-j(k'+h)$, and apparently an essential singularity at $-jk'$. Consequently, the indicated section of the imaginary axis must be excluded from the complex s-plane.

We would like to evaluate the Laplace inversion integral 2.7) by closing the contour in the left half plane and finding the contribution from the various singularities. The contour may only be closed if the integral converges over a large semicircle in the left half plane. Accordingly, the next step is to investigate the behavior of the integrand on this semicircle.

We shall first dispose of the second term in equation 2.13) for $\bar{E}_z$. This contributes to J_z the amount:

4.2) $$J_{z2} = -\int_0^r r'dr'\left[F_E(r') + F_J(r')\right] \cdot \frac{1}{2\pi i}\int_L \frac{jk'ds}{(s+jk')^2} e^{sz} I_n(\gamma r')K_n(\gamma r)$$

$$-\int_r^b r'dr'\left[F_E(r') + F_J(r')\right] \frac{1}{2\pi i}\int_L \frac{jk'ds}{(s+jk')^2} e^{sz} I_n(\gamma r) K_n(\gamma r')$$

The two integrals over s are of the same character, since in each the argument of the I_n function is smaller than the argument of the K_n function. On the large semicircle we have $\gamma = p + O(1/s)$, and $\mathrm{Re}\, p > 0$. Consequently, on the upper quartercircle we apply the asymptotic forms:

4.3) $$e^{sz} I_n(\gamma r')K_n(\gamma r) \rightarrow e^{sz} \frac{e^{pr'}}{\sqrt{2\pi pr'}} \cdot \sqrt{\frac{\pi}{2pr}}\, e^{-pr}$$

$$\rightarrow \frac{1}{2p\sqrt{rr'}}\, e^{sz-p'(r-r')}$$

Since $\mathrm{Re}\, p > 0$, $\mathrm{Re}\, p(r-r')>0$, for $r>r'$. Also, $\mathrm{Re}\, s < 0$. Therefore, the integrand is exponentially small on the upper quartercircle. The same reasoning is valid for the lower quartercircle. Also, the integrals along straight lines connecting the imaginary axis with the contour of integration will also be exponentially small as the lines move to infinity, except at the imaginary axis itself. At the axis, the integrand is of order s^{-3}, and there is no contribution. Accordingly, the integral for J_2 may be closed in the left half-plane.

The treatment of the $A_1 I_n(\gamma r)$ term is more complicated. For large p, the denominator M_1 in A_1 reduces to $G_n(pa)$. The asymptotic form for $G_n(pa)$ is:

4.4) $$G_n(pa) \rightarrow \frac{1}{\pi} e^{2pa}$$

The several terms in the numerator N_1 shall be treated separately, and we shall show that none of them is large enough to cancel the factor e^{-2pa} which arises from $1/M_1$.

We first discuss the order of magnitude of the C,D coefficients. From the definitions 2.5) we obtain, upon using the monotonic properties of the I_n and K_n functions for $|p|$ sufficiently large and Re $p > 0$:

$$4.5)\quad C_1 \rightarrow e^{pb}\ ,\ C_2 \rightarrow e^{-pb},\ C_3 \rightarrow e^{pa},\ C_4 \rightarrow e^{-pa}\ ,\ D_1 \rightarrow e^{pa}\ ,\ D_2 \rightarrow e^{-pa}$$

The bracketed factor multiplying C_1 is of the order $1 + \alpha_1 G_n(pa)e^{-2pb}$, where α_1 is an algebraic function of p. Multiplying this by C_1 gives expressions of the order e^{pb} and e^{2pa-pb}. Both of these lead to exponentially small terms when divided by M_1. Denoting the remaining terms in A_1 by R_2, R_3, R_4, S_1,S_2, we may show directly that R_2 is of the order e^{-pb}, and the other terms are of the order e^{-pa}. Therefore, all the terms in A_1 are exponentially small on the semicircle and the integral may be closed in the left half-plane and replaced by an integral around the singularities of the integrand.

The only possible singularities of the integral are branch points, which are taken care of by the branch cuts of Figure 2, a possible essential singularity at $s = -jk'$, which we shall discuss later, and poles at the zeroes, if any, of M_1. The zeroes of M_1 correspond to the discrete modes of the system, since evaluating the residues leads to terms behaving like $e^{s_i z}$, where s_i is the value of s at the i'th pole p_i of the integrand.

This shows the great advantage of the Laplace transform method. The amplitudes of the discrete modes are given by the residues of the integral at its poles. No matching of solutions in the helical system to input fields need be employed. Instead, the complete description of the matching is given by the coefficients $C_1, \ldots D_2$, which are effectively the Fourier coefficients of the input fields with respect to the modes of the helical system. Since these modes are not orthogonal, the actual solution of the matching equations would be a very difficult task. A partial treatment has been given in papers by Friedman and Malin[4,5].

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[4] B. Friedman and H. Malin, "Excitation of the Traveling Wave Tube", N.Y.U. Mathematics Research Group Report No. TW-12, February, 1950.

[5] B. Friedman and H. Malin, "Orthogonality Properties of the Modes of the Helical Wire Guide", N.Y.U. Mathematics Research Group Report No.TW-13, October, 1950.

The transcendental equation describing the discrete modes of the system is given by setting $M_1 = 0$. It may be written in the form:

$$0 = pb\, I_n(\gamma b) \left[I_n(pb) - G_n(pa)\, K_n(pb) \right] \left[\frac{\gamma}{p} \frac{I_n'(\gamma b)}{I_n(\gamma b)} - \frac{I_n'(pb) - G_n(pa)\, K_n'(pb)}{I_n(pb) - G_n(pa)\, K_n(pb)} \right] \tag*{4.6)}$$

The vanishing of the second bracketed expression in 4.6) leads to the transcendental equation derived by Chu and Jackson[2] and given in simplified form as equation 30) of their paper in the special case $n = 0$. The apparent modes associated with the vanishing of $I_n(\gamma b)$ or $I_n(pb) - G_n(pa)\, K_n(pb)$ are spurious, since for them the second bracket becomes infinite and M_1 does not actually vanish.

The determination of the solutions of equation 4.6), viewed as an equation in p, is a highly complex task. We shall accordingly use a physical argument, which cannot be completely justified. In the absence of the beam, equation 4.6) reduces to $G_n(pa) = 0$. The roots of this simpler equation have been discussed in great detail by Phillips and Malin[6]. We shall assume that in the presence of the beam, each real root of $G_n(pa) = 0$ which corresponds to forward propagation is split into three roots, one real and two complex, exactly as in the case $n = 0$. Since a cubic equation for p may be derived from 4.6) by an expansion to the first power in powers of the beam current, this supposition is reasonably well justified, although a proof that the cubic actually has complex roots has not been given.

In the case $n = 0$, the equation $G_n(pa) = 0$ has one real positive root. It has no other roots in the complex p-plane for $\mathrm{Re}\, p > 0$, though there are roots for $\mathrm{Re}\, p < 0$ in the neighborhood of $pa = \pm jx_n$, where x_n is the n'th root of $J_0(x) = 0$. These roots do not lie in the allowed portion of the p-plane. Hence, for $n = 0$, the equation $M_1 = 0$ has just three roots, of which one is real and positive and the other two are complex conjugate. Denote these by $p_1 = x_1 + iy_1$, $p_2 = x_1 - iy_1$, $p_3 = x_3$.

We now refer to the location of the poles in the complex s-plane. Corresponding to p_1 there will be a pole, s_1 in the fourth quadrant. The value, s_2 of s corresponding to p_2 is in the third quadrant. There are also two poles on the imaginary axis at the conjugate complex values s_{31}, s_{32}. The root at s_1 gives an amplified forward wave; that at s_2 an attenuated forward wave; that at s_{32} a wave that propagates forward without amplification or attenuation; and that at s_{31} a backward propagated wave. The backward wave is usually ignored, but it is required here to satisfy the initial conditions. It might be removed by providing lumped or distributed attenuation, but that is another problem.

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[6] R. S. Phillips and H. Malin, " A Helical Wave Guide II, " N.Y.U., Mathematics Research Group Report No. 170-3, August, 1947.

When the tube is tuned to resonance, the value of s corresponding to the root x_o of $G_o(x) = 0$ coincides with the singularity at $s = -jk'$. For small values of h, the roots s_1, s_2, s_{22} of the system with the beam will lie close to this point in the complex plane. The roots s_1 and s_2, because of their non-vanishing real parts, may be placed outside the branch cut that encircles the section of the imaginary axis from $s = -j(k'-h)$ to $s = -j(k'+h)$. However, s_{32} lies on the imaginary axis, and its position with respect to the singularity should be investigated. However, the wave corresponding to s_{32} is a non-amplified wave, and it may be associated with the bundle of non-amplified waves that will arise from the integration around the branch cut.

The amplified wave arises from the pole at s_1. When we evaluate the residue at this pole, and simplify slightly, there results:

4.7) $$J_1 = \frac{s_1}{(s_1+jk')^2} \quad \frac{N_1(s_1)}{M_1'(s_1)}$$

$$\begin{aligned} N_1(s_1) = {} & \left\{ I_o(p_1b) - G_o(p_1a)\, K_o(p_1b) \right\} C_1/I_o(\gamma_1 b) \\ & - G_o(p_1a)\, C_2 + C_3 + (k \cot \alpha)^2 I_1(p_1a) K_1(p_1a)\, C_4/\, p_1^2\, K_n^2(p_1a) \\ & + jk\cot\alpha\, K_1(p_1a)\, D_1/p_1K_o(p_1a) - jk\cot\alpha\, I_1(p_1a)\, D_2/p_1K_o(p_1a) \\ & + (sE_\theta(0,a) + jkH_r(0,a))/\, p_1^2 \tan\alpha\, K_o(p_1a) \end{aligned}$$

$$M_1'(s_1) = \partial M_1(s)/\partial s \,\Big|_{s=s_1}$$

Since s_1 is the propagation constant of the amplified wave, and M_1 is proportional to the discontinuity in admittance of the system upon crossing the beam radius, we see that the current in the beam is inversely proportional to the derivative of the admittance discontinuity with respect to propagation constant, evaluated where this discontinuity vanishes.

The numerator $N_1(s_1)$ shows how the initial quantities induce current in the beam. We may split N_1 into the sum of three expressions, which are respectively the contributions of initial currents, initial longitudinal electric fields, and initial longitudinal magnetic fields. These expressions are:

$$N_J = \left\{I_0(p_1b)-G_0(p_1a)K_0(p_1b)\right\}\int_0^a r\,dr\,F_J(r)I_0(\tau_1r)/I_0(\tau_1b) \tag{4.8}$$

$$\begin{aligned} N_E = {} & \left\{I_0(p_1b)-G_0(p_1a)K_0(p_1b)\right\}\int_0^a r\,dr\,F_E(r)\,I_0(\tau_1r)/I_0(\tau_1b) \\ & + \int_b^a r\,dr\,F_E(r)\left[I_0(p_1r)-G_0(p_1a)K_0(p_1r)\right] \\ & + (k\cot\alpha)^2\,I_1(p_1a)K_1(p_1a)\int_a^\infty r\,dr\,F_E(r)\,K_0(p_1r)/p_1^2\,K_0^2(p_1a) \end{aligned}$$

$$\begin{aligned} N_M = {} & jk\cot\alpha\left[\int_0^a r\,dr\,F_M(r)\,I_0(p_1r)K_1(p_1a) \right. \\ & \left. + \int_a^\infty r\,dr\,F_M(r)K_0(p_1r)\,I_1(p_1a)\right]/p_1K_0(p_1a) \\ & + \left(s_1E_\theta(0,a)+jkH_r(0,a)\right)/p_1^2\tan\alpha\,K_0(p_1a) \end{aligned}$$

The expression for N_M may be simplified by a judicious integration by parts, which will eliminate the extra non-integral term. This partial integration may be performed in the expression for N_1 for general n. The resulting expression then attains a very neat and cogent form if the incident fields are expressed in rectangular components and if the θ dependence of the incident fields is restored. There results:

4.9)

$$N_M = \frac{jp^2}{2\pi}\int_0^{2\pi} d\theta \int_0^a r\, dr \Big[I_{n-1}(pr) e^{j(n-1)\theta}\{s(E_x - jE_y) + k\,(H_x - jH_y)\}$$

$$- I_{n+1}(pr) e^{j(n+1)\theta}\{s(E_x + jE_y) - k(H_x + jH_y)\}\Big]\, K_n'(pa)/\mu K_n(pa)$$

$$+ \frac{jp^2}{2\pi}\int_0^{2\pi} d\theta \int_a^{\infty} r\, dr \Big[K_{n-1}(pr) e^{j(n-1)\theta}\{s(E_x - jE_y) + k(H_x - jH_y)\}$$

$$- K_{n+1}(pr) e^{j(n+1)\theta}\{s(E_x + jE_y) - k(H_x + jH_y)\}\Big]\, I_n'(pa)/\mu K_n(pa)$$

in which the argument of E_x, etc. is r,θ. Thus, the incident fields are split into circularly polarized waves. The contribution of these circularly polarized waves is then found by multiplying by the appropriate Green's function, as specified by 4.9). From the setup of 4.9), we see that the amplitude of the n'th Fourier coefficient of the current in the beam depends upon the (n+1)'st and (n-1)'st components of incident transverse electric field.

A similar transformation may be performed upon N_E. One of the terms which arises from the integration by parts is proportional to M_1, and hence it does not contribute to the discrete spectrum. It may be associated with the integral 4.2) for J_{z2}. The other terms cancel. Again restoring the θ dependence, the formula for N_E, analogously to 4.9) for N_M becomes:

4.10)

$$N_E = - \frac{\gamma}{2\pi k}\Big[pbK_n(\gamma b)\{I_n'(pb) - G_n(pa)K_n'(pb)\} - \gamma bK_n'(\gamma b)\{I_n(pb) - C_n(pa)K_n(pb)\}\Big]$$

$$\int_0^{2\pi} d\theta \int_0^b r\, dr \Big[I_{n-1}(\gamma r) e^{j(n-1)\theta}\{s(H_x - jH_y) - k(E_x - jE_y)\}$$

$$- I_{n+1}(\gamma r) e^{j(n+1)\theta}\{s(H_x + jH_y) + k(E_x + jE_y)\}\Big]$$

$$- \frac{p}{2\pi k}\int_0^{2\pi} d\theta \int_b^a r\, dr \Big[\{I_{n-1}(pr) + G_n(pa)K_{n-1}(pr)\}\, e^{j(n-1)\theta}\{s(H_x - jH_y) - k(E_x - jE_y)\}$$

$$- \{I_{n+1}(pr) + G_n(pa)K_{n+1}(pr)\} e^{j(n+1)\theta}\{s(H_x + jH_y) + k(E_x + jE_y)\}\Big]$$

$$- \frac{p}{2\pi k}\{I_n(pa) - G_n(pa)K_n(pa)\}\int_0^{2\pi} d\theta \int_a^{\infty} r\, dr \Big[K_{n-1}(pr) e^{j(n-1)\theta}\{s(H_x - jH_y) - k(E_x - jE_y)\}$$

$$- K_{n+1}(pr) e^{j(n+1)\theta}\{s(H_x + jH_y) + k(E_x + jE_y)\}\Big] / K_n(pa).$$

For the discrete spectrum, the coefficient of the first term simplifies to $-\gamma_1\{I_n(p_1b)-G_n(p_1a)K_n(p_1b)\}\,/2\pi k\, I_n(\gamma_1 b)$. Since for the discrete spectrum s is not far from $-jk'$ and accordingly $|s| \gg k$ for slow waves, we are justified in referring to the contribution of N_M as predominantly transverse electric, since in the integrand s multiplies the transverse electric field, while in N_E s multiplies the transverse magnetic field and N_E is predominantly transverse magnetic. Equation 4.10) gives N_E as a function of the transverse input fields. The equivalent representation in terms of the longitudinal electric input field is given by 4.8) in the case $n = 0$. However, 4.10) is useful in showing how the fields break up into circularly polarized components.

Finally, N_J is given by the first line of equation 4.8), where F_J appears in equation 2.11). Now N_J, N_E, and N_M may be interpreted as follows. The incident field, velocity, and a.c. charge density are presumed to be known. The effect of incident velocity and charge is summed up by N_J, which says, form the linear combination F_J, which gives the relative effect of velocity and charge. Multiply by $I_n(\gamma_1 r)/I_n(\gamma_1 b)$, the transfer coefficient giving the current down the tube due to a unit current at the point r, and integrate over r . Then multiply by $jk'\{I_n(p_1b)-G_n(p_1a)K_n(p_1b)\}/(s_1+jk)^2$, and divide by $M_1'(s_1)$ to obtain the actual current in the mode whose z-dependence is as $e^{s_1 z}$. Similarly, N_E and N_M give the effect of incident transverse magnetic and transverse electric fields, respectively, and the appropriate transfer coefficients show what current is generated by unit input field. For values of n other than zero, the equation $G_n(pa) = 0$ may have several roots. However, in general these will not be in step with the beam. Also, the contribution of these higher modes is small if the beam radius is small compared to the wavelength in the input waveguide. We now briefly consider the continuous spectrum, which arises from the integrals over the branch cuts. The integral along the upper and lower sides of the negative real axis may be regarded as a superposition of attenuated waves, since s is real and negative on this contour. Such a superposition will generally have an approximate $1/z$ dependence for large z. It accordingly may be neglected when the length of the helix is large.

The integral over the part of the imaginary axis from $-jk$ to jk is a superposition of fast propagated waves. Because of the boundedness of the integrand along the contour, this integral will be of no lower order than $z^{-1/2}$ for large z, and will be rapidly oscillating in z. The fast waves are not amplified so this integral also may be neglected.

Finally, the integral along the section of the imaginary axis from $-j(k'-h)$ to $-j(k'+h)$ presents a more complicated problem. This is because of the involved nature of the singularity of the integrand at $s = -jk'$. However, this integral also involves only propagated waves, and, although no detailed proof has been given, we shall presume that at large distances down the tube this last integral may also be neglected compared to the amplified wave.

To summarize, we have developed a general theory, which gives the fields down the tube in terms of the input fields, velocity, and charge density. The major result is equation 4.7), which gives the coefficient of the amplified component of current. The application of this theory to particular types of excitation and the comparison of these results with earlier methods will be given in a second report.

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